$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

$$\xi=\xi(x,y),\eta=\eta(x,y)$$

$$\frac{\partial \xi}{\partial x}=a,\frac{\partial \xi}{\partial y}=b,\frac{\partial \eta}{\partial x}=c,\frac{\partial \eta}{\partial y}=d$$

とおく。

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=a\frac{\partial u}{\partial \xi}+c\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial \xi}+c\frac{\partial u}{\partial \eta}\right)$$

$$=a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)+\frac{\partial a}{\partial x}\frac{\partial u}{\partial \xi}+c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)+\frac{\partial c}{\partial x}\frac{\partial u}{\partial \eta}$$

$$=a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)+c\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)+\frac{\partial a}{\partial x}\frac{\partial u}{\partial \xi}+\frac{\partial c}{\partial x}\frac{\partial u}{\partial \eta}$$

第３項、第４項はｕの１階微分なので省略すると

$$\frac{\partial^2 u}{\partial x^2}\to a\left\{a\frac{\partial^2 u}{\partial \xi^2}+c\frac{\partial^2 u}{\partial \xi\partial \eta}\right\}+c\left\{a\frac{\partial^2 u}{\partial \xi\partial \eta}+c\frac{\partial^2 u}{\partial \eta^2}\right\}$$

$$=a^2\frac{\partial^2 u}{\partial \xi^2}+2ac\frac{\partial^2 u}{\partial \xi\partial \eta}+c^2\frac{\partial^2 u}{\partial \eta^2}\quad\cdot\cdot\cdot①$$

同様にして

$$\frac{\partial^2 u}{\partial y^2}\to b^2\frac{\partial^2 u}{\partial \xi^2}+2bd\frac{\partial^2 u}{\partial \xi\partial \eta}+d^2\frac{\partial^2 u}{\partial \eta^2}\quad\cdot\cdot\cdot②$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}b+\frac{\partial u}{\partial \eta}d\right)$$

$$=b\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)+\frac{\partial u}{\partial \xi}\frac{\partial b}{\partial x}+d\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)+\frac{\partial u}{\partial \eta}\frac{\partial d}{\partial x}$$

$$=b\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)+d\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)+\frac{\partial b}{\partial x}\frac{\partial u}{\partial \xi}+\frac{\partial d}{\partial x}\frac{\partial u}{\partial \eta}$$

第３項、第４項はｕの１階微分なので省略すると

$$\frac{\partial^2 u}{\partial x\partial y}\to b\left\{a\frac{\partial^2 u}{\partial \xi^2}+c\frac{\partial^2 u}{\partial \xi\partial \eta}\right\}+d\left\{a\frac{\partial^2 u}{\partial \xi\partial \eta}+c\frac{\partial^2 u}{\partial \eta^2}\right\}$$

$$=ab\frac{\partial^2 u}{\partial \xi^2}+(bc+ad)\frac{\partial^2 u}{\partial \xi\partial \eta}+cd\frac{\partial^2 u}{\partial \eta^2}\quad\cdot\cdot\cdot③$$

①②③より

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}$$

$$=A\left\{a^2\frac{\partial^2 u}{\partial \xi^2}+2ac\frac{\partial^2 u}{\partial \xi\partial \eta}+c^2\frac{\partial^2 u}{\partial \eta^2}\right\}$$

$$+B\left(ab\frac{\partial^2 u}{\partial \xi^2}+(bc+ad)\frac{\partial^2 u}{\partial \xi\partial \eta}+cd\frac{\partial^2 u}{\partial \eta^2}\right)$$

$$+C\left\{b^2\frac{\partial^2 u}{\partial \xi^2}+2bd\frac{\partial^2 u}{\partial \xi\partial \eta}+d^2\frac{\partial^2 u}{\partial \eta^2}\right\}$$

$$=\{Aa^2+Bab+Cb^2\}\frac{\partial^2 u}{\partial \xi^2}+\{2Aac+B(bc+ad)+2Cbd\}\frac{\partial^2 u}{\partial \xi\partial \eta}+\{Ac^2+Bcd+Cd^2\}\frac{\partial^2 u}{\partial \eta^2}$$

よって変換後の係数は

$A'=Aa^2+Bab+Cb^2$

$B'=2Aac+B(bc+ad)+2Cbd$

$C'=Ac^2+Bcd+Cd^2$

なので

$$\begin{aligned}B'^2-4A'C'&=\{2Aac+B(bc+ad)+2Cbd\}^2-4\{Aa^2+Bab+Cb^2\}\{Ac^2+Bcd+Cd^2\}\\&=4A^2a^2c^2+B^2(bc+ad)^2+4C^2b^2d^2+4ABac(bc+ad)+4BCbd(bc+ad)\\&\quad+8ACabcd\\&\quad-4(A^2a^2c^2+ABa^2cd+ACa^2d^2+ABabc^2+B^2abcd+BCabd^2+ACb^2c^2\\&\quad+BCb^2cd+C^2b^2d^2)\\&=(B^2-4AC)(ad-bc)^2\end{aligned}$$